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Exact WKB expansions for some potentials

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Abstract. We find WKB expansions for the spectra of the three-dimensional isotropic harmonic oscillator and Eckart and Morse potentials. It is shown that for all these potentials the series are convergent and, when summed, yield the exact energy spectra. These results support our working hypothesis that the WKB series for the energy spectra are convergent for all one-dimensional solvable potentials and their sums yield the exact energies.

1. Introduction

In this paper we consider the eigenvalue problem for the one-dimensional Schrödinger equation for the list of potentials examined by Rosenzweig and Krieger (1968). Using the method of Fröman and Fröman (1965) they found the exact quantization conditions and energy eigenvalues for the linear and three-dimensional isotropic harmonic oscillators and Coulomb, Pöschl–Teller, Eckart and Morse potentials.

The general exact quantization condition for the one-dimensional Schrödinger equation

$$\left[-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)\right]\psi(r) = E\psi(r) \tag{1}$$

obtained by Fröman and Fröman (1965) is

$$\int_{x'}^{x''} |q(x)|\,\mathrm{d}x = \left(n + \frac{1}{2}\right)\pi - \arg\frac{F_{22}(z,+\infty)}{F_{22}(z,-\infty)} \tag{2}$$

where $q(z)$ and F_{22} are some functions which need to be determined. Rosenzweig and Krieger (1968) showed that for the potentials mentioned above the functions $q(x)$ can be chosen such that $\arg\frac{F_{22}(z,+\infty)}{F_{22}(z,-\infty)} = 0$ and, therefore, the quantization condition has the simple form

$$\int_{x'}^{x''} |q(x)|\,\mathrm{d}x = (n + \tfrac{1}{2})\pi. \tag{3}$$

We will show that the energy eigenvalues for all these potentials can be derived using *the standard WKB series*.

Let us write the wavefunction as

$$\psi(x) = \exp\left\{\frac{\mathrm{i}}{\hbar}\sigma(x)\right\} \tag{4}$$

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where

$$\sigma(x) = \sum_{k=0}^{\infty} \left(\frac{\hbar}{\mathrm{i}}\right)^k \sigma_k(x). \tag{5}$$

If we assume that the potential $V(x)$ is a single-valued analytic function which is real on the real axis, that $V(\pm\infty) = \infty$, that $V(x)$ has a unique minimum on the real axis, that $V(x)$ rises monotonically on both sides of the minimum and for $k > 1$ the integrals

$$\int_{-\infty}^{a} |\sigma_k'(x)|\,\mathrm{d}x \qquad \int_{b}^{+\infty} |\sigma_k'(x)|\,\mathrm{d}x \tag{6}$$

are convergent, when $a < x_1$, $b > x_2$, where $x_1 < x_2$ are the turning points, then, as proven by Fedoryuk (1993), the following asymptotic expansion takes place:

$$\sum_{k=0}^{\infty} \left(\frac{\hbar}{\mathrm{i}}\right)^k \oint_{\gamma} \sigma_k'(x)\,\mathrm{d}x = 2\pi n\hbar \tag{7}$$

where $n \geqslant 0$ is an integer number and γ is a contour surrounding the turning points on the real axis (this formula appears for the first time, apparently, in the work of Dunham (1932)). This relation is an equation with respect to E and using it one can find the asymptotic expansion of the eigenvalues $E_n(\hbar)$. However, it turns out that in some cases the series (8) can be summed exactly (see e.g. Bender *et al* 1977, Robnik and Salasnich 1997a, b, Salasnich and Sattin 1997, Romanovski and Robnik 1999) and it allows us to find not only asymptotic, but exact, eigenvalues for the corresponding potentials. In this paper we will show that this is true for all potentials from the list of Rosenzweig and Krieger (1968). For Coulomb and Pöschl–Teller potentials and for the angular momentum, this was shown in the above-mentioned papers. In this paper we consider the three-dimensional isotropic harmonic oscillator and Eckart and Morse potentials. This paper is a continuation of our previous work (Robnik and Romanovski 2000), in which we studied some general properties of the WKB series, and here we use similar methods. Our motivation and our results, past and present, support our working hypothesis, namely, that the WKB series for a one-dimensional potential converges and gives an exact result for the eigenenergies when the potential is an exactly solvable one. We have no proof of this conjecture so far, but consider it as very important, also in studying potentials that are slightly perturbed from the solvable one.

Note that as shown by Fröman (1966) the odd-order coefficients σ_{2k+1}' are total derivatives. Therefore, we can write the formula (7) in the form

$$\sum_{k=0}^{\infty} \left(\frac{\hbar}{\mathrm{i}}\right)^{2k} \oint_{\gamma} \sigma_{2k}'(x)\,\mathrm{d}x = 2\pi\left(n + \frac{1}{2}\right)\hbar. \tag{8}$$

For the harmonic oscillator and the Morse potential the integrals $\oint_{\gamma} \sigma_{2k}'$ are equal to zero for all $k \geqslant 1$ and, hence, as is well known, the Bohr–Sommerfeld quantization rule is exact in these cases. However, for other potentials the series (8) contains infinitely many terms, and the Bohr–Sommerfeld quantization rule gives us only an approximation of the energy spectra.

Another interesting and important case is the Coulomb potential, which we have solved (Romanovsky and Robnik 1999) using the WKB method. Also, it has been shown by Hainz and Grabert (1999) that a modified WKB series for the Coulomb potential yields the series (8) with only the first two terms different from zero.

2. The Eckart potential

The phase $\sigma(x)$ is a complex function that satisfies the differential equation

$$\sigma'^2(x) + \hbar\sigma''(x) = (V(x) - E) \stackrel{\text{def}}{=} Q(x). \tag{9}$$

Substituting (5) into (9) and comparing like powers of $\hbar$ gives the recursion relation

$$\sigma_0'^2 = 2m(E - V(x)) \qquad \sum_{k=0}^{n} \sigma_k' \sigma_{n-k}' + \sigma_{n-1}'' = 0. \tag{10}$$

We shall denote by $\sigma_k'(x)$ real functions and by $\sigma_k'(z)$ their continuations on the complex plane.

Consider the potential

$$V(x) = \frac{-\lambda \mathrm{e}^{-\alpha x}}{1 - \mathrm{e}^{-\alpha x}} + \frac{b \mathrm{e}^{-\alpha x}}{(1 - \mathrm{e}^{-\alpha x})^2} \tag{11}$$

(where $0 < x < \infty, \alpha, \lambda, b > 0$ and such that there is a potential well), which sometimes is called the Eckart potential (Eckart 1930, Bose 1964).

Let us denote

$$c = 2mE \qquad f = 2mb \qquad g = 2m\lambda \qquad u = \mathrm{e}^{\alpha x} - 1. \tag{12}$$

We cut the complex plane u between the turning points u_1 and u_2 where $0 < u_1 < u_2$, and choose the single-valued branch of the square root, which is negative for large positive u,

$$h(u) = -\sqrt{-cu^2 + (f - g)u + f}. \tag{13}$$

Then

$$\sigma_0'(u) = \frac{\mathrm{i}}{u} h(u) \tag{14}$$

and

$$\sigma_1' = -\frac{\alpha(1+u)((f-g)u + 2f)}{4u\sqrt{c\,u^2 + (g-f)\,u - f}^{\,2}}. \tag{15}$$

We shall show that the coefficients σ_k' of the WKB expansion have the following general form:

$$\sigma_{2s}' = \frac{(u+1)P_{6s-2}(u)}{u\sqrt{c\,u^2 + (g-f)\,u - f}^{\,6s-1}} \tag{16}$$

and

$$\sigma_{2s+1}' = \frac{u(u+1)Q_{6s-1}(u)}{\sqrt{c\,u^2 + (g-f)\,u - f}^{\,6s+2}} \tag{17}$$

where $s > 0$, the degrees of polynomials P_k and Q_k (as functions of u) are not greater than k and P_{6s-2} has the form

$$P_{6s-2} = \frac{1}{2^{2s}} \binom{\frac{1}{2}}{s} (f\alpha)^{2s} \left(1 + \left[3s\left(1 - \frac{g}{f}\right) - 1\right] u + \cdots\right). \tag{18}$$

Here and below we denote by Q_k any polynomial of degree less than or equal to k.

We prove the formulae (16)–(18) by induction on s. It is easy to see that for $s = 1$ the statements are true. Let us suppose that they are proven for all $s < l$ and consider the case $s = l$.

To compute σ_{2l}' using the recurrence formula (10) we note that

$$\frac{\sigma_{2k+1}' \sigma_{2(l-k-1)+1}'}{\sigma_0'} = \frac{u^4(u+1)Q_{6l-7}}{u\sqrt{c\,u^2 + (g-f)\,u - f}^{\,6l-1}} \tag{19}$$

$$\frac{\sigma_1' \sigma_{2(l-1)+1}'}{\sigma_0'} = \frac{u^2(u+1)Q_{6l-5}}{u\sqrt{c\,u^2 + (g-f)\,u - f}^{\,6l-1}} \tag{20}$$

and

$$\frac{\sigma''_{2l-1}}{\sigma'_0} = \frac{u^2(u+1)Q_{6l-4}}{u\sqrt{cu^2+(g-f)u-f}^{\,6l-1}}. \tag{21}$$

Thus we see that the fractions (19)–(21) have the form (16) and the polynomials in the numerators have no influence on the two lowest terms of the numerator of σ'_{2l}.

Therefore

$$\begin{aligned}\sigma'_{2l} &= -\sum_{k=1}^{l-1}\frac{\sigma'_{2k}\sigma'_{2(l-k)}}{2\sigma'_0} + \frac{u^2(u+1)Q_{6l-4}}{u\sqrt{cu^2+(g-f)u-f}^{\,6l-1}}\\ &= -\sum_{k=1}^{l-1}\frac{(u+1)((u+1)P_{6k-2}P_{6(l-k)-2}+u^2Q_{6l-4})}{2u\sqrt{cu^2+(g-f)u-f}^{\,6l-1}}.\end{aligned} \tag{22}$$

Note that the degree of the polynomials $P_{6k-2}P_{6(l-k)-2}$ is less than or equal to $6l-4$. Hence it remains to compute the two lowest terms of the polynomial σ'_{2l}. Observing now that

$$(u+1)P_{6k-2}P_{6(l-k)-2} = \frac{1}{2^{2l}}(f\alpha)^{2l}\binom{\frac{1}{2}}{k}\binom{\frac{1}{2}}{l-k}\left(1+\left[3l\left(1-\frac{g}{f}\right)-1\right]u+\cdots\right) \tag{23}$$

we obtain from (22)

$$\sigma'_{2l} = -\frac{1}{2^{2l+1}}(f\alpha)^{2l}\frac{1+[3l(1-\frac{g}{f})-1]u+u^2Q_{6l-4}}{u\sqrt{cu^2+(g-f)u-f}^{\,6l-1}}\sum_{k=1}^{l-1}\binom{\frac{1}{2}}{k}\binom{\frac{1}{2}}{l-k}. \tag{24}$$

Therefore, taking into account the Vandermonde convolution

$$\sum_{k=0}^{l}\binom{\frac{1}{2}}{k}\binom{\frac{1}{2}}{l-k} = 0 \tag{25}$$

we see that formulae (16), (18) hold.

Direct calculations show that

$$-\frac{\sigma'_1\sigma'_{2l}}{\sigma'_0} - \frac{\sigma''_{2l}}{2\sigma'_0} \qquad \frac{\sigma'_{2k}\sigma'_{2(l-k)+1}}{\sigma'_0} \tag{26}$$

have the form (17). That yields that σ'_{2l+1} has the form (17) as well.

Now we can compute the integrals in (8) using the residue calculus. Taking into account the substitution $u = \mathrm{e}^{\alpha z} - 1$ we obtain

$$\oint \mathrm{d}\sigma_{2k} = \frac{2\pi\mathrm{i}}{\alpha}\left(\mathrm{Res}_0\frac{\sigma'_{2k}}{u+1} + \mathrm{Res}_{-1}\frac{\sigma'_{2k}}{u+1} + \mathrm{Res}_{\infty}\frac{\sigma'_{2k}}{u+1}\right) \tag{27}$$

where $k \geqslant 0$. Because the complex plane is cut between the points u_1 and u_2, where $0 < u_1 < u_2$, and we have chosen the branch defined by (13), we have for $u < u_1$

$$\sigma'_0(u) = \frac{\mathrm{i}}{u}\sqrt{-cu^2+(f-g)u+f} \tag{28}$$

where the root is the arithmetical one.

Therefore

$$\oint_\gamma \mathrm{d}\sigma_0 = 2\pi\sqrt{2m}\left(-\frac{\sqrt{b}}{\alpha} - \frac{\sqrt{-E}}{\alpha} + \frac{\sqrt{-E+\lambda}}{\alpha}\right). \tag{29}$$

In order to simplify the equations we choose our system of units such that $\hbar^2/2m = 1$. Then for $k \geqslant 1$ the formulae (16), (18) and (27) yield

$$\left(\frac{\hbar}{\mathrm{i}}\right)^{2k}\oint_\gamma \mathrm{d}\sigma_{2k} = -2\pi\hbar\left(\frac{1}{4}\right)^k\left(\frac{\sqrt{b}}{\alpha}\right)^{1-2k}\binom{1/2}{k}. \tag{30}$$

Therefore

$$-\sum_{k=0}^{\infty}\hbar\left(\frac{1}{4}\right)^{k}\left(\frac{\sqrt{b}}{\alpha}\right)^{1-2k}\binom{1/2}{k}-\frac{\sqrt{-2Em}}{\alpha}+\frac{\sqrt{2m(-E+\lambda)}}{\alpha}=\hbar\left(n+\frac{1}{2}\right). \tag{31}$$

This gives the condition for the energy eigenvalues

$$-\frac{\sqrt{-E}}{\alpha}-\frac{1}{2}\sqrt{1+\frac{4b}{\alpha^2}}+\frac{\sqrt{-E+\lambda}}{\alpha}=\left(n+\frac{1}{2}\right) \tag{32}$$

where

$$n\leqslant\frac{\sqrt{\lambda}}{\alpha}-\frac{1}{2}-\frac{1}{2}\left(1+\frac{4b}{\alpha^2}\right)^{1/2}. \tag{33}$$

The obtained formula differs from the result by Rosenzweig and Krieger (1968), which states that the energy eigenvalues are

$$E=-\frac{\left(\left(\frac{1}{2}+n+\frac{1}{2}\sqrt{1+\frac{4b}{\alpha^2}}\right)\alpha-\lambda\right)^2}{4\,(1+n+\frac{\sqrt{b}}{\alpha})^2\,\alpha^2} \tag{34}$$

where

$$n\leqslant\frac{\lambda}{\alpha}-\frac{1}{2}-\frac{1}{2}\left(1+\frac{4b}{\alpha^2}\right)^{1/2}. \tag{35}$$

To confirm our formula we shall find the exact solution of the eigenvalue problem. After the substitution $y=\mathrm{e}^{-\alpha x}$ we obtain

$$y^2\,u''(y)+y\,u'(y)+\left(-A^2-\frac{B\,y}{(1-y)^2}+\frac{y\,\beta^2}{1-y}\right)u(y)=0 \tag{36}$$

where

$$A^2=-\frac{2mE}{\hbar^2\alpha^2}\qquad\beta^2=\frac{2m\lambda}{\hbar^2\alpha^2}\qquad B=\frac{2mb}{\hbar^2\alpha^2}. \tag{37}$$

The substitution

$$u(y)=(y-1)^{\frac{1+\sqrt{1+4B}}{2}}\,y^A\,w(y) \tag{38}$$

yields the hypergeometric equation

$$\begin{aligned}2(y-1)\,y\,w''(y)&+2\left(-1-2\,A+\left(2+2\,A+\sqrt{1+4\,B}\right)y\right)w'(y)\\&+\left(1+2\,A+2\,B+(1+2\,A)\sqrt{1+4\,B}-2\,\beta^2\right)w(y)=0\end{aligned} \tag{39}$$

whose general solution is

$$c\,y^{-2\,A}{}_2F_1(\mu-2A,\theta-2A,1-2\,A,y)+d\,{}_2F_1(\mu,\theta,1+2\,A,y) \tag{40}$$

where

$$\mu=\frac{1}{2}+A+\frac{\sqrt{1+4\,B}}{2}-\sqrt{A^2+\beta^2}\qquad\theta=\frac{1}{2}+A+\frac{\sqrt{1+4\,B}}{2}+\sqrt{A^2+\beta^2} \tag{41}$$

c and d are arbitrary constants and, as we see, $\theta>0$. We are interested in the solutions of the boundary value problem

$$u(0)=0\qquad u(1)=0. \tag{42}$$

The first of the equations (42) yields $c = 0$. Hence

$$
\begin{aligned}
u(y) &= d\,(y-1)^{\frac{1+\sqrt{1+4B}}{2}}\, y^A\, {}_2F_1(\mu, \theta, 1+2A, y) \\
&= d\,(y-1)^{\frac{1+\sqrt{1+4B}}{2}}\, y^A\, \frac{\Gamma(1+2A)\Gamma\left(-\sqrt{1+4B}\right)}{\Gamma(1+2A-\mu)\Gamma(1+2A-\theta)} \\
&\quad \times {}_2F_1\left(\mu, \theta, 1+\sqrt{1+4B}, 1-y\right) \\
&\quad + \tilde{d}\,(y-1)^{\frac{1-\sqrt{1+4B}}{2}}\, y^A\, \frac{\Gamma(1+2A)\Gamma\left(\sqrt{1+4B}\right)}{\Gamma(\mu)\Gamma(\theta)} \\
&\quad \times {}_2F_1\left(1+2A-\mu, 1+2A-\theta, 1-\sqrt{1+4B}, 1-y\right).
\end{aligned}
\tag{43}
$$

When $B > 0$ the last term of the sum has a singularity when $y \to 1$, except for the case when μ is a non-positive integer.

Therefore the quantization condition is

$$
-n - \frac{1}{2} = A + \frac{\sqrt{1+4B}}{2} - \sqrt{A^2 + \beta^2} \tag{44}
$$

($n = 0, 1, 2, \ldots$), that coincides with (32). The formula also holds in the case $B = 0$ when the potential degenerates to the Hulthén potential (see, e.g., Flügge 1971).

3. The three-dimensional isotropic harmonic oscillator

Consider the three-dimensional isotropic harmonic oscillator with a degeneracy breaking term. In this case we have the radial equation (1) with

$$
V(x) = \frac{b\hbar^2}{2\,m\,x^2} + \frac{\hbar^2\, l\,(l+1)}{2\,m\,x^2} + \frac{k\,x^2}{2} \tag{45}
$$

and $b, k, m, l > 0$, $0 < x < \infty$. Let us denote

$$
c = \frac{[b + l(l+1)]\hbar^2}{2m} \qquad a = \frac{k}{2} \tag{46}
$$

then the potential has the form

$$
V(x) = \frac{c}{x^2} + a\,x^2. \tag{47}
$$

The first term of the WKB expansion is

$$
\sigma_0' = \sqrt{2}\sqrt{m\left(E - \frac{c}{z^2} - a\,z^2\right)} = \sqrt{2}\sqrt{m\left(E - \frac{b\hbar^2}{2\,m\,z^2} - \frac{\hbar^2\,l\,(l+1)}{2\,m\,z^2} - \frac{k\,z^2}{2}\right)} = \frac{\mathrm{i}}{z}\sqrt{g(z)} \tag{48}
$$

where

$$
g(z) = kmz^4 - 2Emz^2 + b\hbar^2 + \hbar^2\,l\,(l+1) \tag{49}
$$

and

$$
\sigma_1' = \frac{c - a\,z^4}{2\,c\,z - 2\,E\,z^3 + 2\,a\,z^5}. \tag{50}
$$

Let $x_1 < x_2 < 0 < x_3 < x_4$ be the roots of the equation $g(z) = 0$; that is,

$$
g(z) = km(z - x_1)(z - x_2)(z - x_3)(z - x_4). \tag{51}
$$

We cut the complex plane between the points x_1, x_2 and x_3, x_4 and find a single-valued branch $h(z)$ of the function $\sqrt{g(z)}$, such that for $a > x_4$ $h(a) = -\sqrt{g(a)} < 0$. Then we have

$$h(z) = \left|\sqrt{g(z)}\right| \mathrm{e}^{\mathrm{i}\varphi/2} \tag{52}$$

where $\varphi = \Delta_\theta \arg g(z)$ and θ is a curve connecting a and z. From (51) we obtain $\varphi = \sum_{i=1}^{4} \varphi_i$, where $\varphi_i = \Delta_\theta \arg(z - x_i)$. Therefore for $x > x_4$ and $x < x_1$ we obtain

$$h(x) = -\sqrt{g(x)} \tag{53}$$

and when $x_2 < x < x_3$

$$h(x) = \sqrt{g(x)} \tag{54}$$

with the square root being the arithmetical one.

Then (54) yields that for $x_2 < x < x_3$

$$\sigma_0' = \frac{\mathrm{i}}{x}\sqrt{g(x)} \tag{55}$$

and for $x < x_1$ and $x > x_4$ we have according to (53)

$$\sigma_0' = -\frac{\mathrm{i}}{x}\sqrt{g(x)} = -\mathrm{i}x\sqrt{\frac{b\hbar^2 + \hbar^2 l(l+1)}{x^4} - \frac{2mE}{x^2} + km} \tag{56}$$

and, therefore, using the substitution $z^2 = u$ and residue of σ_0' at infinity and the origin, we obtain

$$\oint_\gamma \mathrm{d}\sigma_0 = \pi E\sqrt{\frac{m}{k}} - \pi\hbar\sqrt{b + l + l^2}. \tag{57}$$

To compute the remaining terms of the series (8) one can show (similarly as for the Eckart potential) that the coefficients of the WKB expansion have the following general form:

$$\sigma_{2s}' = \frac{P_{4s}(z^2)}{zm^{s-1/2}\sqrt{E\,z^2 - c - a\,z^4}^{\,6s-1}} \tag{58}$$

and

$$\sigma_{2s+1}' = \frac{zQ_{4s+1}(z^2)}{\sqrt{E\,z^2 - c - a\,z^4}^{\,6s+2}} \tag{59}$$

where $s > 0$, the degrees of polynomials P_k, Q_k (as functions of z^2) are not greater than k and the polynomials P_{4s} have the form

$$P_{4s} = \frac{\sqrt{2}}{2^{3s}}\binom{\frac{1}{2}}{s}(c^{2s} + \cdots). \tag{60}$$

The procedure to prove formulae (58)–(60) is precisely analogous to our proof by induction in section 2.

Hence, taking into account that

$$\oint_\gamma \mathrm{d}\sigma_{2s} = \oint_\gamma \frac{P_{4s}(z^2)}{zm^{s-1/2}\sqrt{E\,z^2 - c - a\,z^4}^{\,6s-1}}\,\mathrm{d}z = \frac{1}{2}\oint_\gamma \frac{P_{4s}(u)}{um^{s-1/2}\sqrt{Eu - c - au^2}^{\,6s-1}}\,\mathrm{d}u \tag{61}$$

we obtain, computing the residue at the origin and infinity,

$$\left(\frac{\hbar}{\mathrm{i}}\right)^{2s}\oint_\gamma \mathrm{d}\sigma_{2s} = -\pi\hbar(b + l + l^2)^{\frac{1}{2}-s}\left(\frac{1}{4}\right)^s\binom{\frac{1}{2}}{s}. \tag{62}$$

This yields

$$E = \hbar\sqrt{\frac{k}{m}}\left(2n + 1 + \left(\frac{1}{4} + b + l(l+1)\right)^{1/2}\right). \tag{63}$$

4. The Morse potential

We now consider the Morse potential

$$V(x) = A\mathrm{e}^{-2\alpha x} - B\mathrm{e}^{-\alpha x}. \tag{64}$$

In this case the formula (10) yields

$$\sigma_0' = \sqrt{2m}\sqrt{E - A\mathrm{e}^{-2\alpha x} + B\mathrm{e}^{-\alpha x}}. \tag{65}$$

We cut the complex plane along the real axis between the turning points x_1, x_2 ($x_1 < x_2$), which are the roots of the equation

$$\frac{A}{\mathrm{e}^{2\alpha x}} - \frac{B}{\mathrm{e}^{\alpha x}} - E = 0 \tag{66}$$

and in such a case we obtain a single-valued function in the cutting plane. We choose the branch such that $-\sqrt{-Ew^2 + A - Bw}$ is negative for large positive w.

It is easy to prove by induction that the coefficients σ_k' have the following form:

$$\sigma_k' = \frac{\mathrm{e}^{-2k\alpha x} P_{2k-1}(\mathrm{e}^{\alpha x})}{\sqrt{E - A\mathrm{e}^{-2\alpha x} + B\mathrm{e}^{-\alpha x}}^{3k-1}} \tag{67}$$

where $k \geqslant 2$; the degrees of polynomials $P_{2k-1}(u)$ (as functions of $u = \mathrm{e}^{\alpha x}$) are not greater than $2k - 1$.

Now we can compute the integrals (8). After the substitution $w = \mathrm{e}^{\alpha x}$ we obtain

$$\oint_\gamma \mathrm{d}\sigma_k = \oint_\gamma \frac{w^{-2k} P_{2k-1}(w)}{\alpha w \sqrt{E - \frac{A}{w^2} + \frac{B}{w}}^{3k-1}} \mathrm{d}w = \oint_\gamma \frac{w^{k-2} P_{2k-1}(w)}{\alpha\sqrt{Ew^2 - A + Bw}^{3k-1}} \mathrm{d}w. \tag{68}$$

The only singularity for the function under the last integral is the infinity. However, it is easily seen that for $k \geqslant 2$

$$\mathrm{Res}_\infty \frac{w^{k-2} P_{2k-1}(w)}{\alpha\sqrt{Ew^2 - A + Bw}^{3k-1}} = 0. \tag{69}$$

Hence

$$\oint_\gamma \mathrm{d}\sigma_k = 0 \tag{70}$$

for all such k and the quantization condition (8) has the form

$$\oint_\gamma \mathrm{d}\sigma_0 = \frac{\sqrt{2m}}{\alpha} \oint_\gamma \frac{\sqrt{Ew^2 - A + Bw}}{w^2} \mathrm{d}w = 2\pi\hbar \left(n + \frac{1}{2}\right). \tag{71}$$

Because

$$\mathrm{Res}_0 \sigma_0' = -\frac{\mathrm{i}B\sqrt{m}}{\sqrt{2A}\,\alpha} \qquad \mathrm{Res}_\infty \sigma_0' = \frac{\mathrm{i}\sqrt{2}\sqrt{-E\,m}}{\alpha} \tag{72}$$

we obtain

$$\frac{B\sqrt{m}}{\sqrt{2\,A}\,\alpha} - \frac{\sqrt{2}\sqrt{-E\,m}}{\alpha} = \hbar\left(n + \frac{1}{2}\right). \tag{73}$$

From this equation we obtain the well known formula

$$E = -\frac{B}{2}\left(\sqrt{\frac{B}{2A}} - \frac{\hbar\,(n + \frac{1}{2})\,\alpha}{\sqrt{B\,m}}\right)^2 \tag{74}$$

and

$$n \leqslant \frac{B}{\alpha\hbar}\sqrt{\frac{m}{2A}} - \frac{1}{2} \tag{75}$$

in accordance with Rosenzweig and Krieger (1968); for the special case $B = 2A$ (see Flügge 1971).

5. Conclusions

In summary, we have demonstrated that for some potentials the quantization condition (8) yields convergent WKB series which give us the exact spectra. The methods for computing these series are similar for all these potentials and consist of two steps: (i) first, we have to find a general formula for the terms σ_k' of the WKB expansions for the phase; (ii) second, we evaluate the contour integrals using the residue calculus.

An important question arising in this connection is what the class of potentials is for which the formula (8) yields the exact spectra. Our experience in this paper and the previous papers by Bender *et al* (1977), Robnik and Salasnich (1997a, b), Romanovski and Robnik (1999) and Salasnich and Sattin (1997) seems to confirm our working hypothesis that the WKB method is exact in all cases of solvable one-dimensional potentials. This has now been confirmed at least for all potentials on the list of Rosenzweig and Krieger (1968). Why exactly this is so is an important unanswered question. Another important issue is the investigation of WKB series for potentials near the solvable cases. This is subject of our future work.

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